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SPRINGBACK IN THERMOVISCOELASTIC CHANNEL SECTIONS

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SUMMARY: When thermoplastic laminated sheets are formed in a mould at high temperature and subsequently cooled and solidified, they undergo distortion from the moulded shape. This is due to the anisotropic thermal properties that cause differential thermal contractions in different directions. For thermoplastic channel sections of uniform thickness with curvilinear orthotropic anisotropy, an analysis given in [1] yielded a simple formula for the change in channel angle in terms of the temperature drop and thermal expansion coefficients in the orthotropic principal directions. In practice it has been found that this formula is very robust, and gives good results for materials that are not usually regarded as elastic. Here it is shown that the analysis extends to linearly viscoelastic materials to yield the same expression for the change in channel angle. The analysis applies to any channel of uniform thickness provided that the orthotropic axes are in the axial, through-thickness and tangential directions. Moreover the main result is even more general, and applies equally to thermoplastic-plastic and thermoviscoplastic solids. The only constitutive requirement is that the strain can be expressed as the sum of a thermal strain and a mechanical strain, and that the mechanical strain vanishes when the stress vanishes.

KEYWORDS: Springback, Thermal Distortion, Dimensional Stability, Fibre Composite laminates, Laminate Forming, Channel Sections, thermoviscoelasticity

INTRODUCTION

'Springback' is an effect that occurs in the processing of fibre-reinforced thermoplastic composite shells. Laminated shells of complicated shape are often formed by pressing stacks of sheets of initially flat, unidirectionally reinforced, but differently oriented, fibre-reinforced thermoplastic matrix material into a mould at a high temperature at which the matrix flows freely. At the forming temperature the material is effectively fluid, although the deformation is constrained by the presence of the fibres. After cooling, solidification and removal from the mould, the product is usually found to have distorted from the shape of the mould. This distortion is due to thermoelastic deformation that accompanies cooling to ambient temperature after solidification at a higher temperature. Because the solid composite material is highly anisotropic, both in its thermal and its mechanical properties, the amount of thermal contraction depends strongly on direction in the material, which results in a change of shape as well as the uniform contraction found in an isotropic material.

The simplest case is that of a channel section, which has single curvature. For linear thermoelastic deformations O'Neill, Rogers and Spencer [1] gave a simple analysis. Considering channel sections of arbitrary cross-section and uniform (but not necessarily small) thickness of linearly thermoelastic material that is curvilinearly orthotropic with orthotropic axes lying in the axial, normal through-thickness and tangential directions (which is the appropriate material symmetry for, among other layups, cross-ply and balanced angle-ply laminates) it was shown in [1] that there exists a stress-free distorted configuration. The principal result is that a channel section which at solidification has a channel angle 2β opens by an angular displacement 2Δβ where

$$\frac{\Delta\beta}{\beta} = -(e_1 - e_2) \tag{1}$$

where e_1 and e_2 are the thermal strains in the radial and tangential directions respectively associated with fall in temperature from the solidification temperature to the ambient temperature, and of course are directly related to the coefficients of thermal expansion in these directions. It has also been shown [2] that this analysis can be extended to finite thermoelastic deformations with similar simple results.

In practice the formula (1) has been found to be very robust, and to give good results for materials that would not normally be regarded as behaving elastically. The purpose of this paper is to provide an explanation for this observation. After a summary of the thermoelastic solution we consider a quite general class of linearly thermoviscoelastic materials with the same geometrical configuration and material symmetry, and show that for these materials also there exists a stress-free solution with exactly the same displacements as in the thermoelastic case. Further generalizations are then described, and it is shown that the same analysis remains valid when the material has thermoelastic-plastic or thermoviscoelastic-plastic response, and also for certain kinds of inhomogeneity. The basic constitutive requirement for (1) to apply is only that the strain should be decomposable into two parts, of which one part depends only on the temperature history, and the other part depends only on the stress history.

TBERMOELASTIC SPRINGBACK

It is convenient first to outline the theory of elastic springback of channel sections as described by O'Neill, Rogers and Spencer [1]. Initially all vector and tensor quantities are referred to a set of cylindrical polar coordinates (*r*, θ, *z*). In this system displacement components are denoted by (*u, v, w*) and the components of the stress tensor σ as

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix}. \tag{2}$$

In the first instance we consider a sector of a circular cylindrical shell of linearly thermoelastic material bounded by the surfaces $r = a$ and $r = b$ and the planes θ = β, θ = -β, as shown in Fig. 1. The material is supposed to have cylindrical orthotropic symmetry with the orthotropic axes coincident with the r, θ, z coordinate curves. This is the appropriate symmetry, for example, for a cross-ply or balanced angle-ply laminated shell. Then the stress-strain relations can be expressed in the form

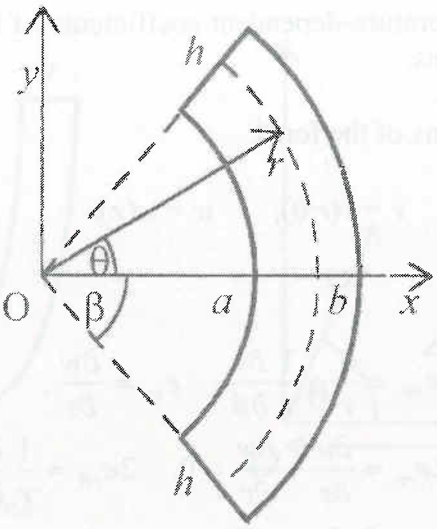

Figure 1 Sector of a circular cylindrical shell

$$\begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix},$$

$$\begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} s_{44}\sigma_{\theta z} \\ s_{55}\sigma_{rz} \\ s_{66}\sigma_{r\theta} \end{bmatrix}, \tag{3}$$

where

$$\begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{r\theta} & e_{\theta\theta} & e_{\theta z} \\ e_{rz} & e_{\theta z} & e_{zz} \end{bmatrix}$$

are components of the infinitesimal strain tensor e, and s_{ij} are the stiffness coefficients that define the elastic mechanical properties of the material. In the general case there are nine independent coefficients s_{ij} for a material with orthotropic symmetry .The principal thermal strains (e_1,e_2,e_3) are given as

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \int_0^T \alpha_1(T)\,dT \\ \int_0^T \alpha_2(T)\,dT \\ \int_0^T \alpha_3(T)\,dT \end{bmatrix} \tag{4}$$

where T is the temperature in excess of a reference temperature at which the undeformed material is stress-free The reference temperature is here taken to be the temperature at which the fluid material solidifies, so that T is negative when the material is in the solid phase. Also

$\alpha_1(T)$, $\alpha_2(T)$, $\alpha_3(T)$ are temperature-dependent coefficients of linear thermal expansion in the principal orthotropic directions.

These equations have solutions of the form

$$u = u(r,\theta), \qquad v = v(r,\theta), \qquad w = w(z), \tag{5}$$

in which case

$$\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad e_{zz} = \frac{\partial w}{\partial z}, \\ 2e_{\theta z} &= \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} = 0, \quad 2e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0, \quad 2e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \end{aligned} \tag{6}$$

The object is to determine zero stress solutions such that $\sigma = \mathbf{0}$. From (3) and (6) this requires

$$e_{rr} = e_1, \quad e_{\theta\theta} = e_2, \quad e_{zz} = e_3, \quad e_{r\theta} = 0. \tag{7}$$

These are four equations for the three displacement components u, v, w, but nevertheless (and remarkably) they have the solution

$$u = re_1, \quad v = -r\theta(e_1 - e_2), \quad w = ze_3. \tag{8}$$

Since there is no stress the equilibrium equations are trivially satisfied.

In this solution the angular displacement of the plane $\theta = const.$ is

$$\Delta\theta = \frac{v}{r} = -\theta(e_1 - e_2), \tag{9}$$

and so for a channel section with sector angle 2β, the increase in channel angle $2\Delta\beta$ is given as

$$\frac{\Delta\beta}{\beta} = -(e_1 - e_2). \tag{10}$$

In practice α_1 is usually considerably larger than α_2, and T is negative, and hence e_2 is greater than e_1, and so the channel angle increases as the material cools. Various other sections can be constructed from straight segments and sectors of circular cylinders; for example the case of a U-section formed from three straight segments and two 90° circular cylinder sectors was described in [1].

This analysis readily extends to channels of arbitrary cross-section, such as that illustrated in Fig. 2. For this case it is convenient to use the quasi-polar coordinates (ξ,ϕ,z) shown in Fig. 3. Suppose that one surface of the section (for definiteness, the inner surface) is defined by a curve C in any normal cross-section of the channel. Then the coordinate ξ denotes distance from C along the outward normal to C, and ϕ is the angle this normal makes with a datum plane, which is taken to be the plane $y = 0$ in a rectangular Cartesian coordinate system (x,y,z)

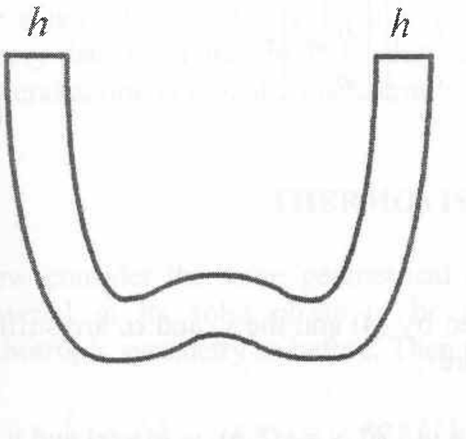

Figure 2 A channel cross-section of arbitrary shape

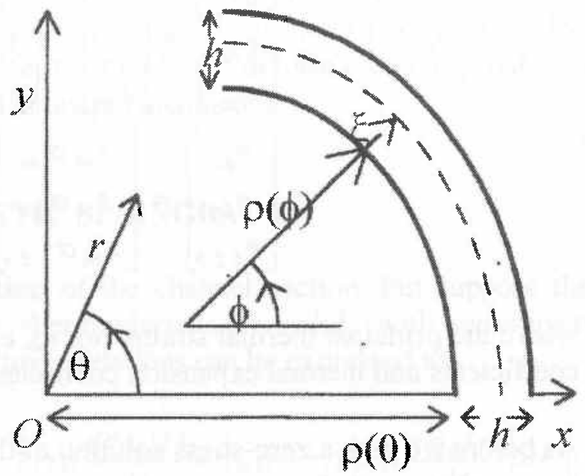

Figure 3 Quasi-polar coordinate system

with its origin at the centre of curvature of C at $\phi = 0$. Then $\xi = 0$ denotes the inner surface of the channel, and for a channel of uniform thickness h, the outer surface is $\xi = h$. If $\rho(\phi)$ is the radius of curvature of C at $(0,\phi,z)$, then (ξ,ϕ,z) are related to (x,y,z) as

$$x = \rho(0) - \int_0^{\phi} \rho(\eta)\sin\eta d\eta + \xi\cos\phi, \quad y = \int_0^{\phi} \rho(\eta)\cos\eta d\eta + \xi\sin\phi, \quad z = z. \tag{11}$$

If ρ is constant, then the channel is a sector of a circular cylinder, and ρ,ϕ,z become conventional cylindrical polar coordinates. We now interpret (u,v,w) as displacement components in the (ξ,ϕ,z) system, and denote stress and strain components in this system as

$$\begin{bmatrix} \sigma_{\xi\xi} & \sigma_{\xi\phi} & \sigma_{\xi z} \\ \sigma_{\xi\phi} & \sigma_{\phi\phi} & \sigma_{\phi z} \\ \sigma_{\xi z} & \sigma_{\phi z} & \sigma_{zz} \end{bmatrix} \quad and \quad \begin{bmatrix} e_{\xi\xi} & e_{\xi\phi} & e_{\xi z} \\ e_{\xi\phi} & e_{\phi\phi} & e_{\phi z} \\ e_{\xi z} & e_{\phi z} & e_{zz} \end{bmatrix} \tag{12}$$

respectively, where

$$e_{\xi\xi} = \frac{\partial u}{\partial \xi}, \quad e_{\phi\phi} = \frac{u}{(\rho+\xi)} + \frac{1}{(\rho+\xi)}\frac{\partial v}{\partial \phi}, \quad e_{zz} = \frac{\partial w}{\partial z},$$
$$2e_{\phi z} = \frac{1}{(\rho+\xi)}\frac{\partial w}{\partial \phi} + \frac{\partial v}{\partial z}, \quad 2e_{\xi z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial \xi}, \quad 2e_{\xi\phi} = \frac{1}{(\rho+\xi)}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial \xi} - \frac{v}{(\rho+\xi)}. \tag{13}$$

There is a minor restriction on the allowed geometry of the section in that it must be such that normals to C do not intersect within the section. The material is supposed to be curvilinear orthotropic with the orthotropic axes coincident with the (ξ,ϕ,z) coordinate curves; again, this is appropriate for a cross-ply or balanced angle-ply laminate. In this case the stress-strain relations are

$$\begin{bmatrix} e_{\xi\xi} \\ e_{\phi\phi} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} \begin{bmatrix} \sigma_{\xi\xi} \\ \sigma_{\phi\phi} \\ \sigma_{zz} \end{bmatrix},$$

$$\begin{bmatrix} e_{\phi z} \\ e_{\xi z} \\ e_{\xi\xi\phi} \end{bmatrix} = \begin{bmatrix} s_{44}\sigma_{\phi z} \\ s_{55}\sigma_{\xi z} \\ s_{66}\sigma_{\xi\xi\phi} \end{bmatrix} \tag{14}$$

where the principal thermal strains (e_1, e_2, e_3) are given by (4) and the s_{ij} and α_i are stiffness coefficients and thermal expansion coefficients as before.

As before we seek a zero-stress solution of the form $u = u(\xi,\phi)$, $v = v(\xi,\phi)$, $w = w(z)$ and it can be seen that $\sigma = \mathbf{0}$ if

$$u = e_1\xi + u_0(\phi), \quad v = -(e_1 - e_2)\{\rho(\phi) + \xi\}\phi + \frac{du_0(\phi)}{d\phi}, \quad w = e_3 z, \tag{15}$$

where $u_0(\phi)$ is the normal displacement of the surface $\xi = 0$ and satisfies the differential equation

$$\frac{d^2u_0(\phi)}{d\phi^2} + u_0 = e_1\rho(\phi) + (e_1 - e_2)\phi\frac{d\rho(\phi)}{d\phi}.$$

This has the solution, to within a rigid body translation

$$u_0(\phi) = \int_0^\phi \sin(\phi - \eta)\{e_1\rho(\eta) + (e_1 - e_2)\eta\cos(\phi - \eta)\}\,d\eta,$$

or, after an integration by parts

$$u_0(\phi) = \int_0^\phi \{e_2\sin(\phi - \eta) + (e_1 - e_2)\eta\cos(\phi - \eta)\}\,\rho(\eta)d\eta, \tag{16}$$

The angular displacement is

$$\frac{\partial v}{\partial \xi} = -(e_1 - e_2)\phi, \tag{17}$$

and so, exactly as for a circular section, the change in channel angle is given by

$$\frac{\Delta\beta}{\beta} = -(e_1 - e_2), \tag{18}$$

where $\alpha = \pm\beta$ at the ends of the section.

These are exact solutions in linear thermoelasticity theory. Unlike, for example, the solution for a heated bimetallic strip, they involve no residual stress in the deformed section. The theory has been extended to the case of finite thermoelastic deformations [2] but this generalization is of mainly academic interest and is not pursued here.

THERMOVISCOELASTIC SPRINGBACK

Now consider the same geometrical configuration of the channel section, but suppose the material in its solid phase to be a linearly thermoviscoelastic solid, with curvilinear orthotropic symmetry as before. Then the constitutive relations can be expressed as

$$\begin{bmatrix} e_{\xi\xi} \\ e_{\phi\phi} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} e_1 + \int_0^t \left\{ J_{11}(t-\tau,T(\tau))\frac{d\sigma_{\xi\xi}(\tau)}{d\tau} + J_{12}(t-\tau,T(\tau))\frac{d\sigma_{\phi\phi}(\tau)}{d\tau} + J_{13}(t-\tau,T(\tau))\frac{d\sigma_{zz}(\tau)}{d\tau} \right\} d\tau \\ e_2 + \int_0^t \left\{ J_{12}(t-\tau,T(\tau))\frac{d\sigma_{\xi\xi}(\tau)}{d\tau} + J_{22}(t-\tau,T(\tau))\frac{d\sigma_{\phi\phi}(\tau)}{d\tau} + J_{23}(t-\tau,T(\tau))\frac{d\sigma_{zz}(\tau)}{d\tau} \right\} d\tau \\ e_3 + \int_0^t \left\{ J_{13}(t-\tau,T(\tau))\frac{d\sigma_{\xi\xi}(\tau)}{d\tau} + J_{23}(t-\tau,T(\tau))\frac{d\sigma_{\phi\phi}(\tau)}{d\tau} + J_{33}(t-\tau,T(\tau))\frac{d\sigma_{zz}(\tau)}{d\tau} \right\} d\tau \end{bmatrix},$$

$$\begin{bmatrix} e_{\phi z} \\ e_{\xi z} \\ e_{\xi\phi} \end{bmatrix} = \begin{bmatrix} \int_0^t J_{44}(t-\tau,T(\tau))\frac{d\sigma_{\phi z}(\tau)}{d\tau} d\tau \\ \int_0^t J_{55}(t-\tau,T(\tau))\frac{d\sigma_{\xi z}(\tau)}{d\tau} d\tau \\ \int_0^t J_{66}(t-\tau,T(\tau))\frac{d\sigma_{\xi\phi}(\tau)}{d\tau} d\tau \end{bmatrix}, \qquad (19)$$

where t denotes the current time, the material has solidified at $t = 0$, and the current strain e(t) depends on the temperature and stress at times τ between $\tau = 0$ and $\tau = t$. The thermal strain (e_1, e_2, e_3) is given in terms of the temperature history as

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \int_0^t \alpha_1(t-\tau,T(\tau))\frac{dT}{d\tau} d\tau \\ \int_0^t \alpha_2(t-\tau,T(\tau))\frac{dT}{d\tau} d\tau \\ \int_0^t \alpha_3(t-\tau,T(\tau))\frac{dT}{d\tau} d\tau \end{bmatrix} \qquad (20)$$

This formulation assumes that the strain is linear with respect to the stress but allows nonlinear dependence on the temperature. Here $\alpha_i(t-\tau, T(\tau))$are time-dependent coefficients of thermal expansion, and $J_{ij}(t-\tau, T(\tau))$ are the anisotropic creep compliances. If the thermal expansion coefficients are independent of time, as is usually to be expected in practical cases, then (20) simplify to become

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \int_0^t \alpha_1(T(\tau))\frac{dT}{d\tau} d\tau \\ \int_0^t \alpha_2(T(\tau))\frac{dT}{d\tau} d\tau \\ \int_0^t \alpha_3(T(\tau))\frac{dT}{d\tau} d\tau \end{bmatrix} \qquad (21)$$

which is equivalent to (4).

We seek solutions in which the stress is zero from $t = 0$ to the current time t. From (19) this requires

$$\begin{bmatrix} e_{\xi\xi} \\ e_{\phi\phi} \\ e_{zz} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad \begin{bmatrix} e_{\phi z} \\ e_{\xi z} \\ e_{\xi\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

By analogy with the themoelastic case, we look for solutions of the form

$$u = u(\xi, \phi, t), \quad v = v(\xi, \phi, t), \quad w = w(z, t). \tag{23}$$

Then $e_{\phi z} = 0$, $e_{\xi z} = 0$, and just as in the thermoelastic case we obtain the solution (15) and (16), the only change being that (e_1, e_2, e_3) are now regarded as functions of t. Thus there exist solutions in which the stress is identically zero throughout the cooling operation, and the channel opening angle is given by (18) exactly as in the thermoelastic solution, and there is no internal stress at any time. There is a requirement that the rate of cooling be sufficiently slow (or the thermal conductivity sufficiently large) for spatial temperature gradients in the material to be negligible.

FURTHER GENERALIZATIONS

It is clear by examination of the above analysis that the underlying reason why it succeeds is the separation of the strain into a thermal part (e_1, e_2, e_3) which depends only on the temperature history, and a mechanical part which depends only on the stress history. For any constitutive equation with the property that the mechanical part of strain is zero when the stress history is zero, the equations for the displacements reduce to (22), and have the solution (15) and (16). Many constitutive equations, besides those for thermoelasticity and thermoviscoelasticity, satisfy this requirement. For example any material with elastic-plastic mechanical response with a yield surface that encloses the origin will undergo neither elastic nor plastic deformation while the stress remains zero, and the thermal strain is then the only contribution to the total strain. The same applies to a material with elastic-viscoplastic response, or indeed a material with purely viscous mechanical response.

Furthermore, it is not necessary for the material to be homogeneous. For example, the solution (15) and (16) for a thermoelastic material depends in no way on the stiffness coefficients s_{ij}, and so these may be any functions (subject to thermodynamic restrictions) of position. Similarly in the viscoelastic case, the creep compliance functions $J_{ij}(t-\tau,\ T(\tau))$ may be arbitrary functions of position, and the same applies to the mechanical response functions in the other cases mentioned above.

However there are restrictions on any permitted inhomogeneity of the thermal expansion coefficients. Zero-stress solutions exist if e_3 (and hence $\alpha_3(1)$) depends in an arbitrary way on the coordinate z, but in general e_1 and e_2 (and hence $\alpha_1(1)$ and $\alpha_2(1)$) must be independent of position for zero-stress solutions to be possible. An interesting exception occurs in the case of a circular cylindrical sector, where if α_1 and α_2 have the forms

$$\alpha_1 = a(T) + b(T)\ln r, \quad \alpha_2 = c(T) + b(T)\ln r \tag{24}$$

and correspondingly

$$e_1 = \int_0^T \{a(T) + b(T)\ln r\} dT, \quad e_2 = \int_0^T \{c(T) + b(T)\ln r\} dT, \tag{25}$$

then it can be verified that the expressions (8), namely

$$u = r\int_0^T \{a(T) + b(T)\ln r\} dT, \quad v = -r\theta \int_0^T \{c(T) + b(T)\ln r\} dT, \tag{26}$$

still give a zero-stress solution for any material in which the stain can be decomposed into a thermal part and a mechanical part.

CONCLUSIONS

It has been shown that the analysis of the deformation of orthotropic channel sections that was described in [1] for linear thermoelastic materials is also valid for a wide variety of mechanical responses, including thermoviscoelastic, thermoelastic-plastic, viscous and thermoviscoplastic behaviour, and moreover allows almost arbitrary mechanical inhomogeneity. The main requirement for the analysis to apply is that the strain response can be decomposed as the sum of a thermal strain, due to thermal expansion or contraction, and a mechanical strain which remains zero when no stress is present. In these circumstances, there exist zero stress solutions independent of the mechanical properties in which the strain is due only to thermal expansion or contraction, and the opening of a channel section due to cooling after solidification is given by the simple formula derived in [1].

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