CONSTITUTIVE EQUATIONS FOR FABRIC-REINFORCED VISCOUS FLUIDS

A.J.M.Spencer

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

ABSTRACT. Constitutive equation are formulated for flow of fabric reinforced composite materials which show linear viscous response at forming temperature.The effect of symmetry of the fabric archititecture is considered. The theory is applied to the analysis of the 'picture-frame' test, and the paradox of different responses in different shearing directions is explained qualitatively.

1.INTRODUCTION

During forming at temperatures above the melting temperature of the matrix, fibre-reinforced composite materials have been successfully modelled as incompressible viscous or viscoelastic fluids reinforced by one or more families of inextensible fibres. In the case of unidirectional reinforcement there is a well-established theory of the mechanical behaviour of such materials[1,2].

Currently there is interest in the case in which the reinforcement is provided by a woven fabric. The corresponding theory for such materials is less well developed, although contributions have been made by Rogers[3], Johnson[4] and McGuinness and Ó Brádaigh[5]. However, there does not seem to be any systematic investigation of the effect of fabric architecture. That this can be considerable is illustrated by experimental results in a 'picture-frame' test described in [5], which showed that for some, but not all, fabrics the shear stress during forming depended on the shear direction, as well as the magnitude and rate of shear. This effect cannot be explained by any theory currently available.

In this paper we formulate a theory for linear viscous reponse that takes proper account of the symmetries of the fabric architecture, and apply this theory to analyse the picture-frame test. It is shown that for some fabrics the theory predicts different behaviour in positive and negative shear in the picture-frame test. These fabrics include the satin weave which was seen in [5] to have this anomolous behaviour.

2. BACKGROUND AND GENERAL THEORY

The deformation and stress are described in a fixed rectangular coordinate system, and all vector and tensor components are components in this system. In a reference configuration, a typical particle has position vector $\mathbf{X}$ (components X_R); at time t the same particle has position vector $\mathbf{x}$ (components x_i). The deformation is thus described by equations

$$\mathbf{x} = \mathbf{x}(X,t), \quad \text{or} \quad x_i = x_i(X_R.t) \qquad (i,R = 1,2,3) \tag{2.1}$$

The velocity $\mathbf{v}$ (components v_i) at $\mathbf{x}$ is $v(x,t)$ and the rate-of deformation tensor $\mathbf{D}$ (components D_{ij}) is defined by

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{2.2}$$

We consider that the composite material consists of a matrix material reinforced by two families of fibres. A continuum theory is employed, in which the fibres are regarded as continuously distributed, with the two fibre directions defined by unit vectors $\mathbf{a}(\mathbf{x},t)$ and $\mathbf{b}(\mathbf{x},t)$ (components $a_i(x_i,t)$ and $b_i(x_i,t)$). The fibres convect with the material and so, in general, **a** and **b** vary both spatially and in time. The rate of change of **a** is given by

$$\frac{Da_i}{Dt} = \frac{\partial a_i}{\partial t} + v_j\frac{\partial a_i}{\partial x_j} = (\delta_{ij} - a_i a_j)a_k\frac{\partial v_j}{\partial x_k} \tag{2.3}$$

where δ_{ij} denotes the Kronecker delta, and here and subsequently repeated suffix summation convention is employed. An analogous relation holds for **b**.

It is assumed that the composite material is incompressible and incapable of extension in the fibre directions. Theories based on incompressibility and fibre inextensibility have been effective in analysing forming processes for uniaxially reinforced composites. The incompressibility condition is

$$D_{ii} = \partial v_i/\partial x_i = 0 \tag{2.4}$$

and the fibre inextensibility conditions are

$$a_i a_j D_{ij} = a_i a_j \partial v_i/\partial x_j = 0, \quad b_i b_j D_{ij} = b_i b_j \partial v_i/\partial x_j = 0, \tag{2.5}$$

If (2.5) hold, then (2.3) simplifies to

$$Da_i/Dt = a_k \partial v_i/\partial x_k. \tag{2.6}$$

The stress tensor is denoted by σ (components σ_{ij}). In quasi-static flows the equilibrium equations

$$\partial\sigma_{ij}/\partial x_j = 0 \tag{2.7}$$

are applicable. For a material subject to kinematic constraints such as incompressibility and inextensibility, the stress can be expressed as the sum of a reaction stress σ_R and an extra-stress τ, where σ_R is a reaction to the constraints and does no work in any deformation conforming to the constraints; for the constraints (2.4) and (2.5)

$$\sigma_R = -p\mathbf{I} + T_a\mathbf{A} + T_b\mathbf{B} \tag{2.8}$$

where p is an arbitrary hydrostatic pressure, T_a and T_b are arbitrary tensions in the fibre directions, **I** is the unit tensor, and **A, B** are tensors with components

$$A_{ij} = a_i a_j, \quad B_{ij} = b_i b_j. \tag{2.9}$$

To complete the formulation a constitutive equation is required for the extra-stress τ. It is assumed that at forming temperatures the composite behaves as an anisotropic viscous fluid, so that **D** is the only kinematic variable involved. The directional properties derive from the presence of the reinforcing fibres. Thus a physically based model can be constructed by making the constitutive assumption

$$\tau = \tau(\mathbf{D},\mathbf{a},\mathbf{b}). \tag{2.10}$$

This relation must be form-invariant under rigid rotations, so τ has to be an isotropic function of its arguments. In addition, it is usual to assume that the senses of **a** and **b** have no significance (the fibres do not have arrows attached to them) and hence τ is even in **a** and in **b**. Thus (2.10) can be replaced by

$$\tau = \tau(\mathbf{D},\mathbf{A},\mathbf{B}). \tag{2.11}$$

The representation of a tensor function of three tensors is an algebraic problem whose solution is known and can be read off from tables[6,7]. In this paper we treat the material as a linear anisotropic viscous fluid (analogous to an isotropic Newtonian fluid) in which τ is linear in **D**. Approximately linear viscous behaviour has been observed in some unidirectionally reinforced composites, but others show prominent non-linearity. Extension to non-linear viscosity and to viscoelasticity is discussed briefly in section 5.

The analogous theory for a linear elastic solid is well developed[8,9]. Rogers[3] drew attention to the correspondence between linear elastic and linear viscous behaviour, with the infinitesimal strain tensor **E** corresponding to the rate-of- deformation tensor **D**. Spencer[9] formulated the constitutive equation for an incompressible linear elastic solid with two families of inextensible fibres (an earlier formulation[8] contains redundant terms). Replacing **E** by **D** in[9] gives a constitutive equation for an incompressible linear viscous fluid similarly reinforced as

$$\sigma = -p\mathbf{I} + T_a\mathbf{A} + T_b\mathbf{B} + 2\eta\mathbf{D} + 2\eta_1(\mathbf{AD} + \mathbf{DA}) + 2\eta_2(\mathbf{BD} + \mathbf{DB}) + 2\eta_3(\mathrm{tr}\,\mathbf{CD})(\mathbf{C} + \mathbf{C}^T). \tag{2.12}$$

Here **C** is the tensor with components $a_i b_j$, tr denotes the trace operation, superscript T represents the transpose, and η, η_1, η_2 and η_3 are viscosities which may be even functions of $\mathbf{a}.\mathbf{b} = \cos 2\phi$, and 2ϕ is the angle between the two families of fibres. Rogers[3], Johnson[4], and McGuinness and Ó Brádaigh[5] gave expressions equivalent to (2.12), but with $\eta_1 = \eta_2$. Setting these coefficients equal results from assuming the two families of fibres to be mehanically equivalent, which, depending on the fabric architecture, may or may not be appropriate. This restriction was made explicit by Rogers[3], but not by the other authors quoted.

For some purposes it is convenient to express the constitutive equation not in terms of the fibre vectors **a** and **b**, but their bisectors **p** and **q**, defined by

$$\mathbf{a} = \mathbf{p}\cos\phi - \mathbf{q}\sin\phi, \quad \mathbf{b} = \mathbf{p}\cos\phi + \mathbf{q}\sin\phi,$$
$$\mathbf{p} = (\mathbf{a} + \mathbf{b})/2\cos\phi, \quad \mathbf{q} = (\mathbf{b} - \mathbf{a})/2\sin\phi. \tag{2.13}$$

and tensors **P**, **Q**, and **M** with components $p_i p_j, q_i q_j$ and $p_i q_j$ respectively, so that

$$\mathbf{A} = \mathbf{P}\cos^2\phi + \mathbf{Q}\sin^2\phi - (\mathbf{M} + \mathbf{M}^T)\sin\phi\cos\phi,$$
$$\mathbf{B} = \mathbf{P}\cos^2\phi + \mathbf{Q}\sin^2\phi + (\mathbf{M} + \mathbf{M}^T)\sin\phi\cos\phi, \tag{2.14}$$
$$\mathbf{C} = \mathbf{P}\cos^2\phi - \mathbf{Q}\sin^2\phi + (\mathbf{M} - \mathbf{M}^T)\sin\phi\cos\phi.$$

This formulation has the advantage that the vectors **p** and **q** are always orthogonal, whereas the angle between **a** and **b** varies during flow. In terms of **P**, **Q** and **M**, (2.12) becomes

$$\begin{aligned}\sigma = {} & -p\mathbf{I} + (T_a + T_b)(\mathbf{P}\cos^2\phi + \mathbf{Q}\sin^2\phi) - (T_a - T_b)(\mathbf{M} + \mathbf{M}^T)\sin\phi\cos\phi + 2\eta\mathbf{D} \\ & + 2(\eta_1 + \eta_2)\{(\mathbf{PD} + \mathbf{DP})\cos^2\phi + (\mathbf{QD} + \mathbf{DQ}\sin^2\phi\} \\ & - 2(\eta_1 - \eta_2)\{(\mathbf{M} + \mathbf{M}^T)\mathbf{D} + \mathbf{D}(\mathbf{M} + \mathbf{M}^T)\}\sin\phi\cos\phi \\ & + 2\eta_3\{(\mathrm{tr}\,\mathbf{PD})\cos^2\phi - (\mathrm{tr}\,\mathbf{QD})\sin^2\phi\}(2\mathbf{P}\cos^2\phi - 2\mathbf{Q}\sin^2\phi).\end{aligned} \tag{2.15}$$

3. THE PICTURE FRAME PARADOX

The picture-frame experiment was developed by McGuinness and Ó Brádaigh[5] to produce a homogeneous time-dependent deformation in uni- or bi-directionally reinforced sheets, as a means to study the rheological behaviour of composite materials in intra-ply shearing. In the experiment an initially square specimen is subjected to a shearing deformation by a linkage of four rigid bars attached to the sides of the specimen. The reinforcing fibres are parallel to the sides of the specimen. The deformation is by stretching along a diagonal of the specimen,

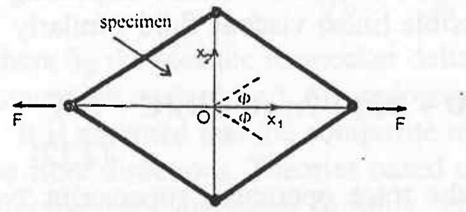

Figure 1. The picture-frame test

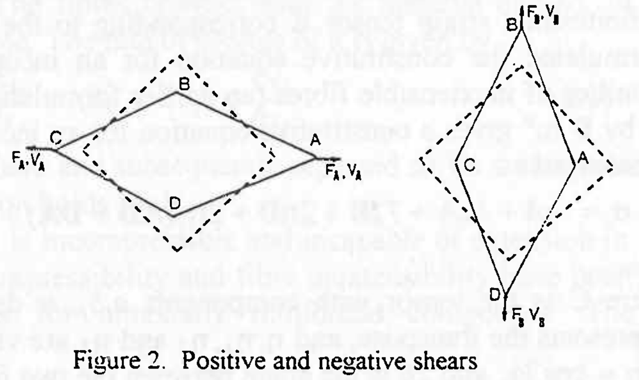

Figure 2. Positive and negative shears

which deforms into a rhombus, as shown schematically in Figure 1. The deformation is not quite one of pure shear, because the area of the sheet is not conserved and so, for an effectively incompressible material, the specimen thickens during the test. An analysis of the deformation was given in [5], but for the present purpose it is convenient to proceed slightly differently.

As shown in Figure 1, the sheet lies in the (x_1, x_2) plane, with the origin at the centre of the square specimen and the axes lying along the diagonals. The fibres are parallel to the sides of the specimen and during deformation are inclined at angles $\pm\phi$ to the x_1–axis. Initially $\phi = \phi_0$ and for an initially square specimen $\phi_0 = \pi/4$. The deformation is described by

$$x_1 = X_1 \frac{\cos\phi}{\cos\phi_0}, \quad x_2 = X_2 \frac{\sin\phi}{\sin\phi_0}, \quad x_3 = X_3 \frac{\sin 2\phi_0}{\sin 2\phi}. \tag{3.1}$$

It can be verified that this deformation satisfies the conditions of incompressibility and fibre inextensibility. The corresponding velocity field is

$$v_1 = -X_1 \frac{\sin\phi}{\cos\phi_0} \frac{D\phi}{Dt} = -x_1 \tan\phi \frac{D\phi}{Dt}, \quad v_2 = X_2 \frac{\cos\phi}{\sin\phi_0} \frac{D\phi}{Dt} = x_2 \tan\phi \frac{D\phi}{Dt},$$

$$v_3 = -2X_3 \frac{\sin 2\phi_0 \cos 2\phi}{\sin^2 2\phi} \frac{D\phi}{Dt} = -2x_3 \cot 2\phi \frac{D\phi}{Dt}, \tag{3.2}$$

and the rate-of-deformation tensor is

$$\mathbf{D} = \begin{bmatrix} -\tan\phi & 0 & 0 \\ 0 & \cot\phi & 0 \\ 0 & 0 & -2\cot 2\phi \end{bmatrix} \frac{D\phi}{Dt} \tag{3.3}$$

This also conforms to the kinematic constraints.

The fibre direction vectors in the picture-frame test are

$$\mathbf{a} = (-\cos\phi, \sin\phi, 0), \quad \mathbf{b} = (\cos\phi, \sin\phi, 0), \tag{3.4}$$

so

$$\mathbf{A} = \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi & 0 \\ -\cos\phi\sin\phi & \sin^2\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.5}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (3.6)$$

Let L denote the length of the sides of the specimen. Then from (3.2) the speeds of the vertices A and B are

$$V_A = -L\sin\phi\, D\phi/Dt, \quad V_B = L\cos\phi\, D\phi/Dt, \qquad (3.7)$$

respectively. By substituting from (3.3), (3.5) and (3.6) into (2.15), the stress is determined as

$$\begin{aligned} \sigma_{11} &= -p + (T_a + T_b)\cos^2\phi - \{2\eta\tan\phi + 2(\eta_1 + \eta_2)\sin 2\phi + 4\eta_3\cos^2\phi\sin 2\phi\}D\phi/Dt, \\ \sigma_{22} &= -p + (T_a + T_b)\sin^2\phi + \{2\eta\cot\phi + 2(\eta_1 + \eta_2)\sin 2\phi + 4\eta_3\sin^2\phi\sin 2\phi\}D\phi/Dt, \\ \sigma_{12} &= -(T_a - T_b)\sin\phi\cos\phi - \{2\eta\cot 2\phi - 2(\eta_1 - \eta_2)\cos 2\phi\sin\phi\}D\phi/Dt, \\ \sigma_{33} &= -p - 4\eta\cot 2\phi\, D\phi/Dt. \end{aligned} \qquad (3.8)$$

If the lateral faces of the sheet are traction-free then $\sigma_{33} = 0$ and p is determined. T_a and T_b are indeterminate unless additional edge boundary conditions are specified.

Suppose the sheet is deformed by forces of magnitude F_A applied in the directions shown in Figure 2a. Following the terminology of McGuinness and Ó Brádaigh[5], this is termed positive shear. By equating the rate of working of the applied forces to the energy dissipation rate, there follows

$$2F_A V_A = (\sigma_{11}D_{11} + \sigma_{22}D_{22} + \sigma_{33}D_{33} + 2\sigma_{12}D_{12})L^2 h, \qquad (3.9)$$

where h is the thickness of the sheet. Using (3.3), (3.7) and (3.8), and noting that the constraints are workless, it follows from (3.9) that

$$F_A = hV_A\{2\eta(\tan^2\phi + \cot^2\phi - 1) + 2(\eta_1 + \eta_2) + \eta_3\sin^2 2\phi\}\,\mathrm{cosec}^2\phi. \qquad (3.10)$$

Alternatively, suppose the deformation is by forces F_B applied as shown in Figure 2b (negative shear, in the terminology of [5]). Then a similar calculation gives

$$F_B = hV_B\{2\eta(\tan^2\phi + \cot^2\phi - 1) + 2(\eta_1 + \eta_2) + \eta_3\sin^2 2\phi\}\sec^2\phi. \qquad (3.11)$$

Now (3.10) and (3.11) become identical if F_B, V_B and ϕ are replaced by F_A, V_A and $\pi/2 - \phi$ respectively. Hence the theory predicts that the material responds identically to positive and negative shears.

The paradox is that for some, but not all of the fabrics which McGuinness and Ó Brádaigh tested in the picture-frame test, there were very different responses to positive and negative shears. For a certain satin weave, the force for given velocity and displacement in positive shear was nearly double the corresponding force in negative shear.

This anomaly cannot be explained by introducing nonlinear viscous or linear or nonlinear viscoelastic behaviour. Any formulation in which the stress depends on the tensors **A** and **B** and kinematic tensors (strain, rate-of deformation, etc.) will predict identical responses to positive and negative shear.

4. RESOLUTION OF THE PARADOX

For a fabric reinforced by inextensible fibres, the mechanical behaviour is governed by the properties of the matrix and the fabric architecture. The paradox cannot be resolved by varying mechanical properties so it is necessary to examine the effects of fabric architecture,

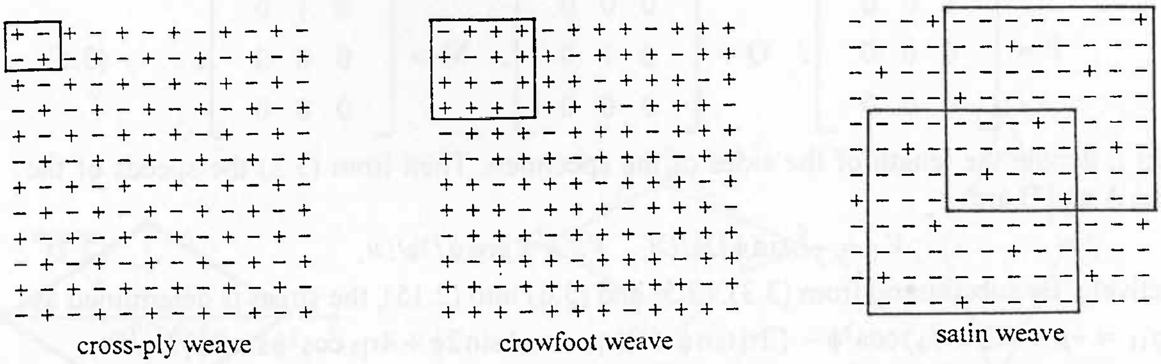

Figure 3. Fabric architectures, showing unit cells

and in particular the symmetries of particular fabrics.

McGuinness and Ó Brádaigh[5] performed experiments for positive and negative shear on two different fabrics; a satin weave and a crowfoot weave. For comparison we also consider a simple cross-ply configuration. These are all illustrated in Figure 3. In this figure the warp is horizontal and the weft is vertical, so that in the reference configuration the vector **a** is directed horizontally from left to right and the vector **b** is upward vertical. The vector **p** is directed diagonally from top left to bottom right, and the vector **q** diagonally from lower left to upper right. A convenient way to visualise the symmetries of the weaves is to consider the intersections between the warp and weft. Accordingly, in figure 3, a '+' denotes an intersection at which the warp passes over the weft. and a '–' a point at which the warp passes under the weft. For each architecture the weave has a unit cell, which is the smallest element that is repeated periodically in the material: for each weave the unit cells are shown in boxes in Figure 3. The unit cell is not unique, and for the satin weave two different unit cells are shown.

The cross-ply configuration has many symmetries. It is unchanged by rotations about the x_3 -axis through $\pi/2$ or π; by reflections in planes normal to the in-plane horizontal and vertical and in planes normal to the bisectors of these directions. It is clear that such a material must respond identically to positive and negative shears. The crowfoot and satin weaves are more complex. Experimentally the crowfoot weave gave the same response in positive and negative shear, but the satin weave gave different responses, so it is appropriate to examine the symmetries of these weaves. Clearly the cross-ply unit cell has all the symmetries mentioned above. The crowfoot weave is symmetric only for reflections in the diagonals, and for rotations through π about the x_3 –axis (which is the product of the two reflections). The upper satin weave unit cell in Figure 3 has only symmetry for reflection in one diagonal, but the lower illustrated satin weave cell is symmetric for reflection in the other diagonal, so we expect the composite material to have both symmetries.

The key to resolving the paradox is the assumption made in Section 2 that the sense of the fibre vectors **a** and **b** is not significant. This is equivalent to assuming reflectional symmetry in planes normal to these vectors. This holds in the cross-ply configuration, so the theory of Section 2 is valid (this configuration also requires $\eta_1 = \eta_2$, so that the constitutive equation proposed in [3] and [4] is applicable). However the other weaves do not have this symmetry. The solution is to revert to (2.10) rather than (2.11), so that the requirement for τ to be even in **a** and in **b** is dropped. Under this generalization all the terms in (2.12) are retained, but examination of tables of tensor functions shows that two additional terms can be admitted, so that the constitutive equation becomes

$$\sigma = -p\mathbf{I} + T_a\mathbf{A} + T_b\mathbf{B} + 2\eta\mathbf{D} + 2\eta_1(\mathbf{AD} + \mathbf{DA}) + 2\eta_2(\mathbf{BD} + \mathbf{DB}) + 2\eta_3(\text{tr}\,\mathbf{CD})(\mathbf{C} + \mathbf{C}^T) + 2\eta_4(\mathbf{CD} + \mathbf{DC}^T) + 2\eta_5(\mathbf{C}^T\mathbf{D} + \mathbf{DC}), \tag{4.1}$$

or, in terms of $\mathbf{P}, \mathbf{Q}$ and $\mathbf{M}$

$$\begin{aligned}\sigma = &-p\mathbf{I} + (T_a + T_b)(\mathbf{P}\cos^2\phi + \mathbf{Q}\sin^2\phi) - (T_a - T_b)(\mathbf{M} + \mathbf{M}^T)\sin\phi\cos\phi + 2\eta\mathbf{D} \\ &+ 2(\eta_1 + \eta_2 + \eta_4 + \eta_5)(\mathbf{PD} + \mathbf{DP})\cos^2\phi + 2(\eta_1 + \eta_2 - \eta_4 - \eta_5)(\mathbf{QD} + \mathbf{DQ})\sin^2\phi \\ &- 2\{(\eta_1 - \eta_2 - \eta_4 + \eta_5)(\mathbf{MD} + \mathbf{DM}^T) \\ &+ (\eta_1 - \eta_2 + \eta_4 - \eta_5)(\mathbf{M}^T\mathbf{D} + \mathbf{DM})\}\sin\phi\cos\phi \\ &+ 2\eta_3\{(\text{tr}\,\mathbf{PD})\cos^2\phi - (\text{tr}\,\mathbf{QD})\sin^2\phi\}(2\mathbf{P}\cos^2\phi - 2\mathbf{Q}\sin^2\phi).\end{aligned} \tag{4.2}$$

For the picture-frame test, with $\mathbf{D}, \mathbf{P}, \mathbf{Q}$ and $\mathbf{M}$ given by (3.3) and (3.6), this becomes

$$\begin{aligned}\sigma_{11} &= -p + (T_a + T_b)\cos^2\phi - \{2\eta\tan\phi + 2(\eta_1 + \eta_2 + \eta_4 + \eta_5)\sin 2\phi \\ &\quad + 4\eta_3\cos^2\phi\sin 2\phi\}D\phi/Dt, \\ \sigma_{22} &= -p + (T_a + T_b)\sin^2\phi + \{2\eta\cot\phi + 2(\eta_1 + \eta_2 - \eta_4 - \eta_5)\sin 2\phi \\ &\quad + 4\eta_3\sin^2\phi\sin 2\phi\}D\phi/Dt, \\ \sigma_{12} &= -(T_a - T_b)\sin\phi\cos\phi - 2\{(\eta_1 - \eta_2)\cos 2\phi - (\eta_4 - \eta_5)\sin\phi\cos\phi\}D\phi/Dt, \\ \sigma_{33} &= -p - 4\eta\cot 2\phi\, D\phi/Dt.\end{aligned} \tag{4.3}$$

By calculations similar to those used in Section 3, there follows, for positive and negative shears respectively

$$\begin{aligned}F_A &= hV_A\{2\eta(\tan^2\phi + \cot^2\phi - 1) + 2(\eta_1 + \eta_2) - (\eta_4 + \eta_5)\cos 2\phi + \eta_3\sin^2 2\phi\}\,\text{cosec}^2\phi, \\ F_B &= hV_B\{2\eta(\tan^2\phi + \cot^2\phi - 1) + 2(\eta_1 + \eta_2) - (\eta_4 + \eta_5)\cos 2\phi + \eta_3\sin^2 2\phi\}\sec^2\phi.\end{aligned} \tag{4.4}$$

Because $\cos(\pi - 2\phi) = -\cos 2\phi$, these expressions are not unchanged if F_B, V_B and ϕ are replaced by F_A, V_A and $\pi/2 - \phi$ respectively, unless $\eta_4 + \eta_5 = 0$, in which case (3.10) and (3.11) are recovered. If $\eta_4 + \eta_5$ is not zero, the responses to positive and negative shears are different. Hence the picture frame paradox is explained if the more general constitutive equation (4.1) is adopted.

This formulation takes no account of any symmetries of the material. Both satin and crowfoot weaves are symmetric for reflections in planes normal to the **p** and **q** directions. Therefore the constitutive equation must be unchanged if **p** is replaced by **-p**, or **q** is replaced by **-q**, or both interchanges are made simultaneously. This is equivalent to requiring (4.2) to be unchanged if **M** is replaced by **-M**. This can only be satisfied if the coefficients of $\mathbf{MD} + \mathbf{DM}^T$ and $\mathbf{M}^T\mathbf{D} + \mathbf{DM}$ in (4.2) are zero, which implies that

$$\eta_1 = \eta_2, \quad \eta_4 = \eta_5. \tag{4.5}$$

For the satin and crowfoot weave materials, these substitutions should be made in the constitutive equations (4.1) and (4.2), and also in (4.3) and (4.4). It is noted that four viscosity coefficients, η, η_1, η_3 and η_4 are needed to characterize the linear viscous behaviour of these materials. The coefficient η_4 can, in principle, be measured by observing the difference between responses to positive and negative shear.

Experimentally, it was found[5] that for a crowfoot weave (but not for a satin weave) $\eta_4 = 0$. There does not seem to be any symmetry argument to account for this observation. It may be significant that the symmetry of the crowfoot weave is more apparent than that of the satin weave, but we see no obvious way in which this distinction can be made precise.

5. DISCUSSION AND CONCLUSIONS

A properly invariant constitutive equation has been formulated for a linearly viscous fabric-reinforced material. This theory accounts qualitatively for the different responses of a satin-weave fabric to positive and to negative shears in the picture-frame test. A query remains in that the theory does not explain the experimental observation that the coefficient η_4 is zero for a crowfoot weave fabric. It may be that this coefficient is too small to be detected in the experiment, or there may be some fundamental reason that requires it to vanish.

This study has been confined to linear viscous behavior, but the same principles can be applied to analyse non-linear viscous response. However the general non-linear viscous theory will involve many response functions, whose experimental determination is unlikely to be practicable, unless further simplifying assumptions are made. For example, a relatively simple theory can be established by allowing the viscosity coefficients to be powers of rhe rate-of-deformation invariant $\sqrt{(D_{ij}D_{ij})}$. Such a power-law relation was found in [5] to fit picture-frame test data quite well in some cases.

Similar remarks apply, but more forcibly, to viscoelastic modelling. For some viscoelastic formulations the same principles can be applied, but the results will be complicated.

REFERENCES

1. B.D. Hull, T.G. Rogers and A.J.M. Spencer. Theoretical analysis of forming flows of continuous-fibre-resin systems. In 'Flow and Rheology in Polymer Composites Manufacturing' (ed. S.G. Advani) 203-256, Elsevier, Amsterdam, 1994

2. C.M. Ó Brádaigh. Sheet forming of composite materials. In 'Flow and Rheology in Polymer Composites Manufacturing' (ed. S.G. Advani) 517-569, Elsevier, Amsterdam, 1994

3. T.G. Rogers. Rheological characterisation of anisotropic materials. Composites 20 (1989) 21-27

4. A.F. Johnson. Rheological model for the forming of fabric reinforced thermoplastic sheets. Composites Manufacturing 6 (1995) 153-160

5. G.B. McGuinness and C.M. Ó Brádaigh. Development of rheological models for forming flows and picture-frame shear testing of fabric reinforced thermoplastic sheets. J. Non-Newtonian Fluid Mech. 73 (1997) 1-28

6. A.J.M. Spencer. Theory of invariants. In 'Continuum Physics' (ed. A.C. Eringen) 240-353, Academic Press, New York, 1971

7. Q-S. Zheng. Theory of representations for tensor functions. Appl.Mech.Rev. 47 (1994) 554-587

8. A.J.M. Spencer. Deformations of fibre-reinforced materials. Clarendon Press, Oxford, 1972

9. A.J.M. Spencer. Continuum theory of the mechanics of fibre-reinforced composites, Chap.1. CISM Courses and Lectures No.282, Springer, Wien, 1984